Name:

Student ID:

History of Mathematics (Math 4123)
Midterm Exam

Professor Paul Bailey
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The examination contains five problems which are worth 20 points each. The extra credit problem is worth 20 additional points. You may use your book and any notes you may have. If you have any questions about the meaning of any of the words or notation on the test, please ask.

Prob 1	Prob 2	Prob 3	Prob 4	Prob 5	ExCred	Total Score

Problem 1. (Ancient Numeral Systems)
Compute the indicated values. Our system is referred to as *Hindu-Arabic*.

(a) Greek Ciphered/Multiplicative System

The Greeks used a combination of the ciphered and multiplicative numeral systems.

Name	Symbol	Arabic	Name	Symbol	Arabic	Name	Symbol	Arabic
alpha	α	1	iota	ι	10	rho	ρ	100
beta	β	2	kappa	κ	20	sigma	σ	200
gamma	γ	3	lambda	λ	30	tau	τ	300
delta	δ	4	mu	μ	40	upsilon	υ	400
epsilon	ϵ	5	nu	ν	50	phi	ϕ	500
digamma	g	6	xi	ξ	60	chi	χ	600
zeta	ζ	7	omicron	o	70	psi	ψ	700
eta	η	8	pi	π	80	omega	ω	800
theta	θ	9	koppa	k	90	sampi	s	900

$$x' = 10^3 x \quad \text{where } \alpha \leq x \leq \theta \qquad x\mathsf{M} = 10^4 x \quad \text{where } \alpha \leq x \leq s$$

(i) Convert $\phi\mathsf{M}\theta\mathsf{M}\delta'\xi$ from Greek to Hindu-Arabic;

(ii) Convert 50403 from Hindu-Arabic to Greek;

(iii) Compute $\xi\mathsf{M}\omega\pi\alpha + \chi\mathsf{M}\eta'\xi$, expressed in Greek.

(b) Mayan Simple/Quasi-Vigesimal System

The Mayans used a combination of the simple base 5 and positional base 20 system, with an exception of base 18 in position 2.

Symbol	Arabic	Symbol	Arabic	Symbol	Arabic	Symbol	Arabic
𝋠	0	|	5	||	10	|||	15
·	1	·|	6	·||	11	·|||	16
:	2	:|	7	:||	12	:|||	17
⋮	3	⋮|	8	⋮||	13	⋮|||	18
⁞	4	⁞|	9	⁞||	14	⁞|||	19

(i) Convert ⋮ ·|| ⋮||| from Mayan to Hindu-Arabic;

(ii) Convert 50403 from Hindu-Arabic to Mayan;

(iii) Compute :|| :||| + ⋮ ·|| :| , expressed in Mayan.

Problem 2. (Pythagorean Triples)
The Babylonians generated tables of Pythagorean triples (a, b, c) such that a is sexagesimally regular. Euclid's *Elements* supplied a technique for computing Pythagorean triples using the equations

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2.$$

Diophantus proved that this produces *all* Pythagorean triples.

Thus the following function generates Pythagorean triples:

$$\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N} \quad \text{by} \quad \phi(u, v) = (2uv, u^2 - v^2, u^2 + v^2).$$

Set

$$S = \{n \in \mathbb{N} \mid \ 1 \leq n \leq 10 \text{ and } n \text{ is decimally regular}\};$$
$$U = \{(u, v) \in S \times S \mid \ v < u \text{ and } \gcd(u, v) = 1\}.$$

(a) Find S.

(b) Find U.

(c) Find $\phi(U)$.

Problem 3. (Regular Solids)
The *regular solids* were studied by the Pythagoreans, the Platonists, and Euclid.

(a) List the regular solids. State the type of regular polygon from which each solid is constructed. Find the number of faces F, the number of edges E, and the number of vertices V. Compute $F - E + V$.

(b) Luca Pacioli (1509) used three intersecting golden rectangles to construct a regular solid whose faces are equilateral triangles with sides of length one. Use this construction to find the radius of a sphere in which such a solid can be transcribed.

Problem 4. (Diophantine Geometry)
A *rational curve* is the set of solutions to a polynomial equation in two variables whose coefficients are rational numbers. A *rational point* on a curve is a solution whose coordinates are rational numbers.

Diophantus (Alexandria, 2nd century A.D.) realized that, given two rational points on a cubic curve, the slope between them would be rational, and so the third point of intersection between the line and the curve would produce another rational point.

Consider the curve given by the equation

$$y^2 = x^3 - 4x + 9.$$

By trying small values for x, find four rational points on this curve. Select two points such that the slope of the line between them is 3. Compute this line. Intersect this line with the curve to find two additional rational points.

Problem 5. (Congruence)
Euclid's *Elements* contains a description of the Euclidean algorithm for find x, y such that

$$mx + ny = \gcd(m, n).$$

The proof of the *Chinese Remainder Theorem* uses this fact to produce solutions to systems of congruences of the form

$$a \equiv c \pmod{m};$$
$$b \equiv c \pmod{n}.$$

Let $m = 17$, $n = 37$, $a = 7$, and $b = 11$.

(a) Find x and y such that $mx + ny = 1$.

(b) Find c with $0 \leq c < mn$ such that $a \equiv c \pmod{m}$ and $b \equiv c \pmod{n}$.

Problem 6. Constructibility [Extra Credit]
Let A be a set of points in a plane $\mathcal{P}$. Let $\mathcal{L}(A)$ be the set of all lines in $\mathcal{P}$ which pass through at least two points in A, and let $\mathcal{C}(A)$ be the set of all circles in $\mathcal{P}$ pass through a point in A and whose center is a different point in A. Let $\mathcal{O}(A) = \mathcal{L}(A) \cup \mathcal{C}(A)$. Define

$$S(A) = \{z \in \mathcal{P} \mid z \in O_1 \cap O_2 \text{ for some } O_1, O_2 \in \mathcal{O}(A)\}.$$

(a) If A contains one point, how large is $S(A)$?

(b) If A contains two points, how large is $S(A)$?

(c) If A contains three collinear equally spaced points, how large is $S(A)$?

(d) If A contains three collinear unequally spaced points, how large is $S(A)$?

(e) If A contains the vertices of an equilateral triangle, how large is $S(A)$?

(f) If A contains the vertices of an acute isosceles triangle, how large is $S(A)$?

(g) If A contains the vertices of an obtuse isosceles triangle, how large is $S(A)$?

Include a drawing to justify each case.